Worcester County Mathematics League

Varsity Meet 3
January 8, 2014

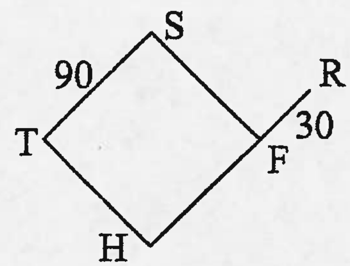
COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Varsity Meet 3 – January 8, 2014
Round 1: Similarity and Pythagorean Theorem

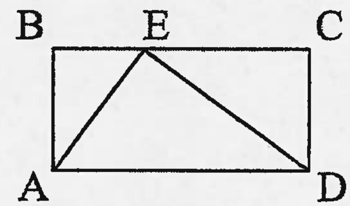
All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

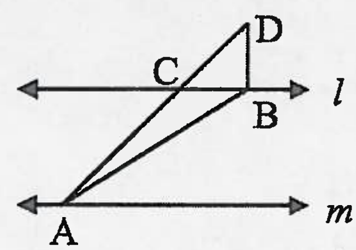
1. A baseball diamond is a square 90 feet on each side. The right fielder picks up the ball 30 feet behind first base (at point R) and throws it to the third baseman at T. In feet, what is the distance from R to T?



2. Given that $AE = 3$, $DE = 4$, and $AD = 5$, find the area of rectangle $ABCD$.



3. In the diagram, $l \parallel m$, $BC = 1$, $CD = 2$, and $AC = 3$. If points A, C, and D are collinear and $\angle CBD$ is a right angle, find AB.



ANSWERS

(1 pt.) 1. ______________________ feet

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

Varsity Meet 3 – January 8, 2014
Round 2: Algebra I

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. If 4 is added to 5 times a number, the result is the same as when 8 is subtracted from twice the number. Find the number.

2. Find the value of m such that $x - 3$ is a factor of $4x^2 - 6x + m$.

3. Suppose that y is a function of x defined as $y(x) = 2x^3 + Ax^2 + Bx - 5$. If $y = 3$ when $x = 2$ and $y = -37$ when $x = -2$, find $A + B$.

ANSWERS

(1 pt.) 1. __________________

(2 pts.) 2. __________________

(3 pts.) 3. __________________

Varsity Meet 3 – January 8, 2014
Round 3: Functions

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Let $f(x)$ be a linear function. If $f(5) = -5$ and $f(-25) = -25$, find $f(2)$.

2. Let $f(a) = \dfrac{a+9b}{a-b}$. Find the value of $\dfrac{a^3}{b^3}$ if $f(a+b) = 6$.

3. If $f(x) = x^3 + 3x^2 + 3x + 5$, find the exact value of $f^{-1}(45)$.

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

Varsity Meet 3 – January 8, 2014
Round 4: Combinatorics

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. If 20 houses in a community all required direct lines to one another in order to have telephone service, how many lines would be necessary?

2. Find the integer N such that $(3!)(5!)(7!) = N!$.
   ($x!$ denotes x factorial.)

3. Suppose a packet of candies contains between 9 and 14 pieces (inclusive), and that the candies come in 3 different colors. How many distinguishable packets are possible?

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. ____________________

Varsity Meet 3 – January 8, 2014
Round 5: Analytic Geometry

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Given that the point $(2,3)$ is the midpoint of the line segment joining $(-4,-2)$ and (x,y), find the sum $x+y$.

2. If $(2014,b)$ and $(a,2014)$ are two points on the line $y=\frac{1}{4}x+17$, what is the value of $\frac{2014-a}{2014-b}$?

3. The center of a circle is on the line $2x-y-1=0$. The circle passes through the points $(2,-1)$ and $(-2,0)$. Find the coordinates of the center.

ANSWERS

(1 pt.) 1. ____________________

(2 pts.) 2. ____________________

(3 pts.) 3. (__________ , __________)

Varsity Meet 3 – January 8, 2014
TEAM ROUND

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (2 POINTS EACH)

APPROVED CALCULATORS ALLOWED

1. Book A has 125 fewer pages than Book B, and Book C has twice as many pages as Book A. Find the number of pages in Book B if Book C has 150 more pages than Book B.

2. Let $f(x) = \dfrac{x-1}{x+1}$. Find $f^5(2014) \equiv f(f(f(f(f(2014)))))$.

3. Nine-sixteenths of $8\frac{1}{3}\%$ of 384 is what **percentage** of 28.8?

4. An ellipse has foci $(2,3)$ and $(8,3)$. Its minor axis is 8 units long. If the equation for the ellipse is written in the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$ with integer coefficients and $A > 0$, find the sum $A + C + D + E + F$.

5. Factor completely: $(x^2 - 9)(x + 2) + (x - 2)(x + 3) - (x + 3)$

6. If $2x - 3y - z = 0$ and $x + 3y - 14z = 0$ with $z \neq 0$, determine the numerical value of
$$\frac{x^2 + 3xy}{y^2 + z^2}.$$

7. If $\dfrac{{}_{100}P_r}{{}_{100}C_r} = 3628800$, find r.

8. The front of a train enters a 3.5-mile tunnel traveling at 20 mph. That speed decreases uniformly (that is, linearly) with respect to time to 10 mph as the front exits the tunnel, and it remains at 10 mph thereafter. The elapsed time from when the front engine entered the tunnel until the <u>last car</u> left the tunnel is 18 minutes. Determine the length of the train in miles.

9. In right triangle ABC, $m\angle C = 90°$. If $AB = 3x + 4$, $BC = 4x - 8$, and $AC = 2x + 1$, find x.

Varsity Meet 3 – January 8, 2014
TEAM ROUND ANSWER SHEET

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (2 POINTS EACH)

1. ____________________ pages

2. ____________________

3. ____________________ %

4. ____________________

5. ____________________

6. ____________________

7. ____________________

8. ____________________ miles

9. ____________________

Varsity Meet 3 – January 30, 2013
ANSWERS

ROUND 1
(St. John's, Doherty, Quaboag)

1. 150 feet
2. 12
3. $\sqrt{13}$

ROUND 2
(Leicester, Bromfield, St. John's)

1. -4
2. -18
3. -1

ROUND 3
(Tantasqua, Quaboag, Westborough)

1. -7
2. 8
3. $\sqrt[3]{41}-1$ or <u>exact</u> equivalent
   ($-1+41^{1/3}$, etc.)

ROUND 4
(Tahanto, Westborough, QSC)

1. 190
2. 10
3. 515

ROUND 5
(South, Northbridge, Doherty)

1. 16
2. -4
3. $(-1/4,-3/2)$ or equivalent
   ($(-0.25,-1\frac{1}{2})$, etc.)

TEAM ROUND
(Shepherd Hill, AMSA Charter, Assabet Valley, Leicester, Bromfield, Tahanto, Worc Acad, Doherty, Clinton)

1. 400
2. $\dfrac{2013}{2015}\approx 0.999$
3. $62.5=62\frac{1}{2}=125/2$
4. -44
5. $(x+3)^2(x-3)$ or equivalent
   $[(x+3)(x+3)(x-3)$, etc.$]$
6. 7
7. 10
8. $2/3=0.\overline{6}\approx 0.667$
9. 7

Varsity Meet 3 – January 30, 2013
FULL SOLUTIONS

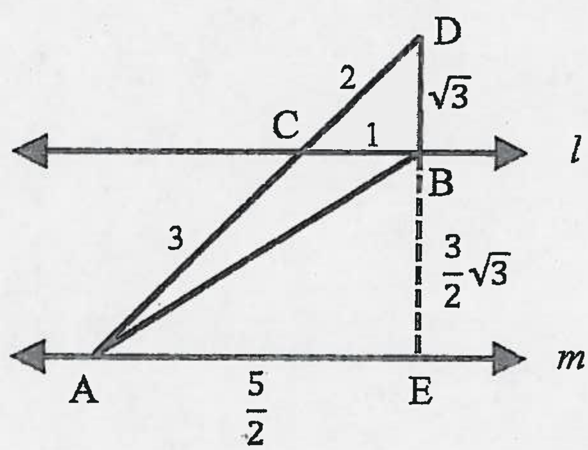
ROUND 1

1. We have that $TH = 90$, $HR = 120$, and $\angle H$ is a right angle. Recognize this as a 3-4-5 right triangle multiplied through by a factor of 30. Therefore, the length of the hypotenuse $\overline{TR}$ is $\boxed{150}$.

2. **METHOD I:** Drop an altitude from E to $\overline{AD}$. By similar triangles (AA similarity, since $m\angle AED = 90°$ from the 3-4-5 triangle), the altitude has length 2.4. Therefore, the area of the rectangle is $5 \cdot 2.4 = \boxed{12}$.

   METHOD II: Since $\triangle AED$ and rectangle $ABCD$ share the same base and height, the area of the rectangle is twice the area of the triangle. The triangle, with sides 3, 4, and 5, is a right triangle with area 6. Therefore, the rectangle has area $6 \cdot 2 = \boxed{12}$.

3. Extend $\overline{DB}$ downwards to meet m at E, as shown in the diagram:



Since $CD = 2$ and $BC = 1$, the Pythagorean Theorem gives $BD = \sqrt{3}$. Also, by AA, $\triangle CBD \sim \triangle AED$ with side length ratio $2:5$. Therefore, $AE = 5/2$ and $DE = \frac{5}{2}\sqrt{3}$. Thus $BE = \frac{3}{2}\sqrt{3}$ and so $AB = \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{3}{2}\sqrt{3}\right)^2} = \sqrt{\frac{25}{4} + \frac{27}{4}} = \boxed{\sqrt{13}}$.

ROUND 2

1. We have $5x+4=2x-8$, so $x=\boxed{-4}$.

2. If $x-3$ is a factor of the polynomial, plugging in $x=3$ will give zero. Therefore:

$$\begin{aligned}4\cdot 3^2-6\cdot 3+m &= 0\\ 36-18+m &= 0\\ m &= \boxed{-18}.\end{aligned}$$

3. Plug in the two (x,y) pairs to find a system of equations:

$$\begin{aligned}4A+2B &= -8\\ 4A-2B &= -16\end{aligned}$$

Solving the system gives $(A,B)=(-3,2)$, so $A+B=\boxed{-1}$.

ROUND 3

1. Since f is a linear function, find the slope of the line that represents $y=f(x)$. The slope is $m=\dfrac{-25-(-5)}{-25-5}=\dfrac{2}{3}$. Pick one of the points to solve for the intercept, and find that $f(x)=\dfrac{2}{3}x-\dfrac{25}{3}$. Plug in $x=2$ to find that $f(2)=\boxed{-7}$.

2. Plug $a+b$ in to the function definition to find that $6=f(a+b)=\dfrac{a+10b}{a}=1+10\dfrac{b}{a}$. Therefore, $\dfrac{a}{b}=2$ and $\dfrac{a^3}{b^3}=\boxed{8}$.

3. Notice that $f(x)=x^3+3x^2+3x+5=(x+1)^3+4$. Therefore, $f^{-1}(x)=\sqrt[3]{x-4}-1$. Plugging in, $f^{-1}(45)=\boxed{\sqrt[3]{41}-1}$.

ROUND 4

1. A line is needed between each pair of houses. There are $\dbinom{20}{2}=\dfrac{20\cdot 19}{2}=\boxed{190}$ such pairs.

2. **METHOD I:** Separate out the factors of the smaller factorials:

$$\begin{aligned}(3!)(5!)(7!) &= 3\cdot 2\cdot 5\cdot 4\cdot 3\cdot 2\cdot (7!)\\ &= (7!)\cdot(4\cdot 2)\cdot(3\cdot 3)\cdot(5\cdot 2)\\ &= (7!)\cdot 8\cdot 9\cdot 10\\ &= 10!\end{aligned}$$

Therefore, $N = \boxed{10}$.

METHOD II: Since 7! is the largest factorial on the left side, $N > 7$. There is no factor of 11 on the left side, and since 11 is prime, $N < 11$. The factor of 5! on the left contributes a factor of 5, which means that $N \geq 10$. Therefore, N must be $\boxed{10}$.

3. First, we use the "Stars and Bars" method of counting. For the case of 9 candies in the bag with 3 colors, represent the candies with nine stars and use two bars to divide the selection into 3 colors. A possible configuration is

$$\star\star\star|\star\star|\star\star\star\star$$

which represents 3 of the first color, 2 of the second color, and 4 of the third color. Another allowed configuration is

$$\star\star\star\star\star||\star\star\star\star$$

which represents 5 of the first color, 0 of the second color, and 4 of the third color. Therefore, for 9 candies in the bag, the problem becomes a question of how many ways there are to arrange the two bars among 11 objects (stars and bars combined).

In the problem, the bags can contain anywhere from 9 to 14 candies. Therefore, the answer is

$$\binom{11}{2}+\binom{12}{2}+\binom{13}{2}+\binom{14}{2}+\binom{15}{2}+\binom{16}{2}.$$

Since $\binom{r}{r}+\binom{r+1}{r}+\binom{r+2}{r}+\dots+\binom{n}{r}=\binom{n+1}{r+1}$ (prove it using Pascal's Triangle), this can be more easily summed as

$$\sum_{n=2}^{16}\binom{n}{2}-\sum_{n=2}^{10}\binom{n}{2}=\binom{17}{3}-\binom{11}{3}=\boxed{515}.$$

(*Of course, it is not* necessary *to use this summation.*)

ROUND 5

1. Since $(2,3)$ is the midpoint, the coordinates of the endpoints must sum to $(4,6)$. Therefore, $(x,y) = (8,8)$ and $x + y = \boxed{16}$.

2. The slope of the line 1/4 is equal to $\dfrac{\Delta y}{\Delta x} = \dfrac{2014-b}{a-2014}$. Invert and multiply by -1 to find that $\dfrac{2014-a}{2014-b} = \boxed{-4}$.

3. If the circle passes through the points $(2,-1)$ and $(-2,0)$, the center must be equidistant from those points. The slope of the line joining those points is $-1/4$, so the line containing all points equidistant to those two has slope 4 and passes through the midpoint, $(0,-1/2)$. This line is $y = 4x - \dfrac{1}{2}$.

   We are also given that the center of the circle lies on the line $2x - y - 1 = 0$. The intersection of these two lines is at the point $\boxed{(-1/4, -3/2)}$.

TEAM ROUND

1. We have the three equations

$$\begin{aligned} A + 125 &= B \\ C &= 2A \\ C &= B + 150 \end{aligned}$$

   Therefore, $2(B-125) = 2A = C = B + 150$. Hence $B = \boxed{400}$.

2. Since $f(x) = \dfrac{x-1}{x+1}$, then

$$\begin{aligned} f\Big(f(x)\Big) &= \frac{\dfrac{x-1}{x+1} - 1}{\dfrac{x-1}{x+1} + 1} \\ &= \frac{(x-1)-(x+1)}{(x-1)+(x+1)} \\ &= \frac{-2}{2x} \\ &= -1/x. \end{aligned}$$

   From this, $f^4(x) = x$. Therefore, $f^5(2014) = f(2014) = \boxed{\dfrac{2013}{2015}}$.

3. We have

$$\begin{aligned} \frac{9}{16} \cdot \frac{1}{12} \cdot 384 &= x\% \cdot 28.8 \\ 18 &= x\% \cdot 28.8 \\ \frac{18}{28.8} &= \frac{x}{100} \\ \boxed{62.5} &= x \end{aligned}$$

4. The ellipse with foci at $(2,3)$ and $(8,3)$ has center $(5,3)$ and major axis along the x-axis (parallel to the line containing the foci). Therefore, the minor axis is along the y-axis. We are given that the minor axis has length 8, so the endpoints of the minor axis are at $(5,-1)$ and $(5,7)$.

   By the definition of an ellipse, the length of the major axis is the sum of the distances to each focus point from any point along the ellipse. Pick the point $(5,7)$ on the ellipse. The sum of the distances to the foci are $2\sqrt{3^2+4^2} = 10$.

   Therefore, the equation for the ellipse can be written

$$\frac{(x-5)^2}{5^2} + \frac{(y-3)^2}{4^2} = 1.$$

   Rearrange into standard form:

$$\begin{aligned} 16(x-5)^2 + 25(y-3)^2 &= 400 \\ 16x^2 - 160x + 400 + 25y^2 - 150y + 225 &= 400 \\ 16x^2 + 25y^2 - 160x - 150y + 225 &= 0. \end{aligned}$$

   The sum of the coefficients is $16 + 25 - 160 - 150 + 225 = \boxed{-44}$.

5. Recognize the difference of squares:

$$\begin{aligned} & (x^2-9)(x+2) + (x-2)(x+3) - (x+3) \\ = & (x+3)\Big[(x-3)(x+2) + (x-3)\Big] \\ = & (x+3)\Big[(x-3)(x+3)\Big] \\ = & \boxed{(x+3)^2(x-3)}. \end{aligned}$$

6. When solving the system, eliminate z to find that $3x = 5y$. Eliminate y to find that $3x = 15z$. Therefore, $(x, y, z) = (15k, 9k, 3k)$ for some constant $k \neq 0$ (since we are given that $z \neq 0$).

   In the fraction to be evaluated, every term is quadratic. Therefore, all of the factors of k will cancel. Thus

$$\begin{aligned} \frac{x^2 + 3xy}{y^2 + z^2} &= \frac{15^2 + 3 \cdot 15 \cdot 9}{9^2 + 3^2} \\ &= \boxed{7}. \end{aligned}$$

7. From the formulas for combinations and permutations,

$$\frac{{}_{100}P_r}{{}_{100}C_r} = r!$$

   Recognize $3628800 = 10!$, so $r = \boxed{10}$.

8. Since the speed of the train decreases linearly with <u>time</u> (and not with <u>distance</u>), the average speed of the train in the tunnel is 15 mph. Since the tunnel is 3.5 miles long, the train spends $\frac{3.5}{15} \cdot 60 = 14$ minutes inside the tunnel.

   Therefore, at 10 mph, it takes the length of the train 18 minutes − 14 minutes = 4 minutes to pass. Thus the length of the train is $\frac{4}{60}$ hr · 10 mph = $\boxed{\frac{2}{3}}$ miles.

9. Use the Pythagorean Theorem on the side lengths.

$$\begin{aligned} (2x+1)^2 + (4x-8)^2 &= (3x+4)^2 \\ 4x^2 + 4x + 1 + 16x^2 - 64x + 64 &= 9x^2 + 24x + 16 \\ 11x^2 - 84x + 49 &= 0 \\ (11x-7)(x-7) &= 0 \\ x &= 7/11 \text{ or } 7 \end{aligned}$$

   Of these possibilities, only $x = \boxed{7}$ gives all positive side lengths when plugged back in to the original triangle.